MAT 5190 Hwk #6. Prof. M. Alvo

on 29

7.1 The MLE is a function of x : it is $\hat{\theta} = 1$ if $x = 0$ or 1

on 3 $= 2$ or 3 if $x = 2$

$= 3$ if $x = 3$ or 4

7.6 a) $f(x_1 \ldots x_n|\theta) = \theta^n (\prod x_i)^{-2}$ $\min x_i \geq \theta$

on 3

$\therefore T = \min X_i$ is sufficient

b) $\frac{\partial}{\partial\theta} \ln f(x_1 \ldots x_n|\theta)$ does not exist ; we use a graphic approach.

The likelihood increases as a function of $\theta \Rightarrow T$ is the MLE

c) $\mu_1 = E X = \int_\theta^\infty \theta x^{-1} dx$ does not exist $\Rightarrow$ the method fails

7.10 $f(x|\alpha,\beta) = \frac{\alpha}{\beta^\alpha} x^{\alpha-1}$ $0 < x < \beta$

on 5 a) $f(x_1 \ldots x_n|\alpha,\beta) = \frac{\alpha^n}{\beta^{n\alpha}} (\prod x_i)^{\alpha-1}$ $0 < \max x_i < \beta$

$\Rightarrow T = [\prod X_i, \max X_i]$ is suff. for (α,β).

b) $\ln f(x_1 \ldots x_n|\alpha,\beta) = n \ln\alpha - n\alpha \ln\beta + (\alpha-1) \ln \prod x_i$

For fixed β, $\frac{\partial \ln f}{\partial\alpha} = 0 \Rightarrow \hat{\alpha} = \frac{1}{\ln\beta - \frac{1}{n}\ln\prod x_i}$

Also f is decreasing in β. Hence the MLE of β is the $\max X_i$

Also $\hat{\alpha}$ is the MLE of α since $\frac{\partial^2 \ln f}{\partial\alpha^2} = -\frac{n}{\alpha^2} < 0$ implying $\hat{\alpha}$ is a global max.

c) $\hat{\beta} = \max x_i = 25$; $\hat{\alpha} = 12.7$

7.20 a) $f(y_1 \ldots y_n) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp - \sum [y_i - \beta x_i]^2 / 2\sigma^2$

$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp - \frac{[\sum y_i^2 - 2\beta \sum x_i y_i + \beta \sum x_i^2]}{2\sigma^2}$

Hence $T = [\sum Y_i^2, \sum x_i Y_i]$ is sufficient for (β, σ^2). The $\{x_i\}$ are fixed.

b) $\frac{\partial}{\partial \beta} \ln f(y_1 \cdots y_n) = \frac{\sum x_i(y_i - \beta x_i)}{\sigma^2} = 0 \Rightarrow \hat\beta = \frac{\sum x_i Y_i}{\sum x_i^2}$

To show $\hat\beta$ is unbiased, $E\hat\beta = \frac{\sum x_i E Y_i}{\sum x_i^2} = \beta \frac{\sum x_i^2}{\sum x_i^2} = \beta$

$Var\,\hat\beta = \frac{\sum x_i^2 \, Var\, Y_i}{(\sum x_i^2)^2} = \frac{\sigma^2}{\sum x_i^2}$

(3)

$\therefore \hat\beta \sim N\left(\beta, \frac{\sigma^2}{\sum x_i^2}\right).$

Now

$V\left(\frac{\sum Y_i}{\sum x_i}\right) = \frac{n\sigma^2}{n^2 \bar x^2} = \frac{\sigma^2}{n \bar x^2} > \frac{\sigma^2}{\sum x_i^2} = Var\,\hat\beta$ since

$\sum(x_i - \bar x)^2 = \sum x_i^2 - n\bar x^2 > 0$

7.25 p 359 a) $X \mid \theta \sim n(\theta, \sigma^2)$ $\quad \theta \sim n(\mu, \tau^2)$

$\therefore$ the marginal of X is obtained by integrating θ in the joint density i.e

$g(x) = \int \frac{1}{\sqrt{2\pi}\,\sigma} \exp -(x-\theta)^2/2\sigma^2 \cdot \frac{1}{\sqrt{2\pi}\,\tau} \exp -(\theta-\mu)^2/2\tau^2 \, d\theta$

keep only non constants $\sim \int \exp -(x^2 - 2\theta x + \theta^2)/2\sigma^2 \cdot \exp -(\theta^2 - 2\mu\theta + \mu^2)/2\tau^2 \, d\theta$

$\sim \exp -\frac{x^2}{2\sigma^2} \cdot \exp -\mu^2/2\tau^2 \int \exp -\frac{1}{2}\left[\theta^2\left(\frac{1}{\sigma^2} + \frac{1}{\tau^2}\right) - 2\theta\left(\frac{x}{\sigma^2} + \frac{\mu}{\tau^2}\right)\right] d\theta$

$\sim \exp -\frac{x^2}{2\sigma^2} \cdot \exp -\frac{\mu^2}{2\tau^2} \cdot \exp \frac{(x/\sigma^2 + \mu/\tau^2)^2}{2(1/\sigma^2 + \frac{1}{\tau^2})}$

$\sim \exp -\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2\tau^2(1/\sigma^2 + \frac{1}{\tau^2})}$ which we recognize as

$N(\mu, \sigma^2 + \tau^2)$

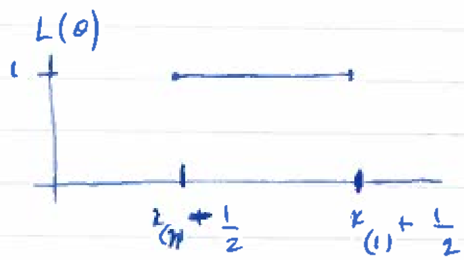
b) In general, $X_i \mid \theta_i$ are indep. $\Longrightarrow$

$$\int \prod f(x_i \mid \theta_i)\, p(\theta_i)\; d\theta_i = \prod \left[\int f(x_i \mid \theta_i)\, p(\theta_i)\, d\theta_i \right]$$

$$= g(x_1) \cdots g(x_n) \text{ and hence}$$

the x's are independent marginally.

Hand out a) We have the likelihood $L(\theta) = \begin{cases} 1 & \theta - \frac{1}{2} < X_{(1)} < X_{(n)} < \theta + \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$



$\therefore\ \hat{\theta}$ is any value in the interval $\left[X_{(n)} - \frac{1}{2} , X_{(1)} + \frac{1}{2} \right]$

b) θ is a location parameter. Hence there exists a r.v. Z such that $X = Z + (\theta - \frac{1}{2})$ and $Z \sim$ unif on $(0, 1)$.

Hence

$$\frac{X_{(1)} + X_{(n)}}{2} = \frac{Z_{(1)} + Z_{(n)}}{2} + \left(\theta - \frac{1}{2}\right)$$

$$= V + \left(\theta - \frac{1}{2}\right).$$

We know $V \sim f_V(v) = \begin{cases} n\,(2v)^{n-1} & 0 < v < \frac{1}{2} \\ n\,[2(1-v)]^{n-1} & \frac{1}{2} < v < 1 \end{cases}$

$$\frac{E[V]}{n\, 2^{n-1}} = \int_0^{1/2} v^n\, dv + \int_{1/2}^{1} v\,(1-v)^{n-1}\, dv = \frac{1}{2^{n+1}(n+1)} + \dots \text{ etc...}$$

$\boxed{\text{or}}$ f_V is symmetric about $1/2$ and has a peak at $1/2$. Hence $E[V] = \frac{1}{2}$ (p. 231)

$$\therefore\ E[X] = E[V] + \theta - \frac{1}{2} = \theta$$

(b continued) To prove consistency we need to show

$$P(|Z_{(n)}-1| \geq \varepsilon) \to 0 \quad \text{as } n \to \infty$$

Note $P(|Z_{(n)}-1| \geq \varepsilon) = P(Z_{(n)} \geq 1+\varepsilon) + P(Z_{(n)} \leq 1-\varepsilon)$

$$= 0 + (1-\varepsilon)^n \to 0 \quad \text{as } n \to \infty \quad 1 > \varepsilon > 0$$

Also $P(|Z_{(1)}-0| \geq \varepsilon) = P(Z_{(1)} \geq 0+\varepsilon) + P(Z_{(1)} \leq 0-\varepsilon)$

$$= \varepsilon^n + 0 \to 0 \quad \text{as } n \to \infty \quad 1 > \varepsilon > 0$$

Hence $Z_{(n)} \xrightarrow[p]{} 1$ and $Z_{(1)} \xrightarrow[p]{} 0$ implying

$$V = \frac{Z_{(n)} + Z_{(1)}}{2} \xrightarrow[p]{} \frac{1}{2}$$

$$\therefore \quad \frac{X_{(n)} + X_{(1)}}{2} = V + \theta - \frac{1}{2} \xrightarrow[p]{} \theta .$$

c) $\hat{\theta}_2 = X_{(n)} - \frac{1}{2} = \left[Z_{(n)} + \theta - \frac{1}{2}\right] - \frac{1}{2} \xrightarrow[p]{} \left[1 + \theta - \frac{1}{2}\right] - \frac{1}{2} = \theta$

$\therefore$ $\hat{\theta}_2$ is consistent

$$MSE(\hat{\theta}_2) = Var[\hat{\theta}_2] + \{E[\hat{\theta}_2 - \theta]\}^2$$

$$Var[\hat{\theta}_2] = Var[Z_{(n)} + \theta - 1] = Var[Z_{(n)}]$$

$$E[\hat{\theta}_2 - \theta] = E[Z_{(n)} - 1]$$

Now $Z_{(n)}$ has density $f_{Z_{(n)}}(u) = n u^{n-1}, \quad 0 < u < 1$

$$\therefore \quad E[Z_{(n)}] = \frac{n}{n+1}, \quad E[Z^2_{(n)}] = \frac{n}{n+2}, \quad Var[Z_{(n)}] = \frac{n}{(n+2)(n+1)^2}$$

$$\therefore MSE(\hat{\theta}_2) = \frac{n}{(n+2)(n+1)^2} + \frac{1}{(n+1)^2} = \frac{2}{(n+1)(n+2)}$$

d) $MSE(\hat{\theta}_1) = Var[\hat{\theta}_1] + \{E(\hat{\theta}_1 - \theta)\}^2$

Since $\hat{\theta}_1 = V + \theta - \frac{1}{2}$, $\quad Var[\hat{\theta}_1] = Var[V]$, $E[\hat{\theta}_1 - \theta] = E[V - \frac{1}{2}]$

From b) $E[V - \frac{1}{2}] = 0$; hence $\hat{\theta}_1$ is unbiased.

We know $f_V(v) = \begin{cases} n(2v)^{n-1}, & 0 < v \le \frac{1}{2} \\ n(2(1-v))^{n-1} & \frac{1}{2} < v < 1. \end{cases}$

$$\frac{E[V^2]}{n2^{n-1}} = \int_0^{1/2} v^2 v^{n-1}\,dv + \int_{1/2}^1 v^2(1-v)^{n-1}\,dv$$

$$= \int_0^{1/2} v^{n+1}\,dv + \int_0^{1/2} (1-u)^2 u^{n-1}\,du$$ on setting $u = 1-v$ in the second

$$= 2\int_0^{1/2} v^{n+1}\,dv + \int_0^{1/2} u^{n-1}\,du - 2\int_0^{1/2} u^n\,du$$

$$= \frac{1}{(n+2)2^{n+1}} + \frac{1}{n2^n} - \frac{1}{(n+1)2^n}$$

$$\therefore E[V^2] = \frac{n}{4(n+2)} + \frac{1}{2(n+1)}$$

$$Var[V] = E[V^2] - \{E[V]\}^2 = \frac{n}{4(n+2)} + \frac{1}{2(n+1)} - \frac{1}{4}$$

$$= \frac{1}{2(n+1)(n+2)}$$

Comparing

$$MSE(\hat{\theta}_2) - MSE(\hat{\theta}_1) = \frac{2}{(n+1)(n+2)} - \frac{1}{2(n+1)(n+2)} = \frac{3}{2(n+1)(n+2)} > 0$$

$$\therefore MSE(\hat{\theta}_2) > MSE(\hat{\theta}_1)$$

but asymptotically the difference goes to 0.